Transform Methods for Linear Systems

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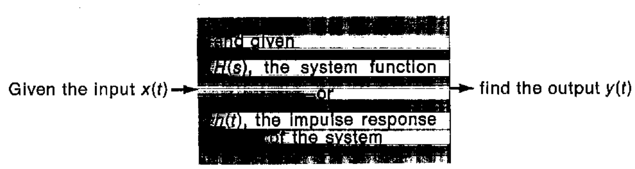
This is a highly condensed collection of reference material on transform methods. Formulas for direct and inverse Fourier, Laplace, and Z-transforms are given in compact tabular form, with examples of how they are used to find system responses for several types of inputs.

Transforms are powerful tools for solving linear time-invariant system problems, such as



Depending upon the system and the excitation, the most convenient transforms to use might be Fourier transforms, one- or two-sided Laplace transforms, or one- or two-sided Z-transforms. These transforms are treated in many textbooks (such as references 1 and 3 in Bibliography), and no attempt has been made to give an exhaustive treatment here. To the author's knowledge, however, there is nothing in the literature equivalent to Table I, which is a comprehensive collection of the defining formulas for all of these transforms and their inverses. The arrangement and notation of this table are designed to demonstrate the similarities and interrelationships between the different transforms. A feature of the treatment of inverses is a logical progression from the fundamental inverse formulas to their equivalents in terms of complex variables (contour integrals, residues, Laurent series).

Table II applies transform methods to the problem of finding the output of a linear system for five types of input signals. In each case the transform used is the most appropriate one for the type of input.

These tables can be used as a reference by anyone working in the field of linear systems or — along with a textbook — by students of linear system theory. As an aid to the student, Table III lists the impedances of the basic network elements R, L, and C for the five classes of inputs.

Classes of Inputs

In the tables and discussion that follow, five classes of inputs are considered. They are:

i) A damped sinusoidal input $x(t) = Ae^{-\sigma_0 t}\cos(\omega_0 t + \phi_0)$

ii) A periodic input $x(t) \doteq \sum_{n=0}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$

iii) An aperiodic input which has a Fourier or a Laplace transform.

iv) A sampled input $x^*(t)$ known only at specified points.

v) A random input, where $x(t)$ is a member of an ergodic random process for which the autocorrelation function $R_x(\tau)$ and its Fourier transform $S_x(j\omega)$ are known.

As an example of the use of the transforms, the aperiodic case (iii) will be discussed in detail. The sampled case (iv) is similar to the aperiodic case except that Z-transforms are used instead of Laplace transforms. The first two cases, i.e., periodic and damped sinusoidal inputs, are well known.

For random inputs (case v) the problem can be restated as follows: Given the input autocorrelation function $R_x(\tau)$ and the system function, find the output autocorrelation function $R_y(\tau)$. With this modification, the problem is similar to the aperiodic case.

Inverses by Complex Variable Methods

The problem of finding the output of a linear system whose input and system function are known may be regarded as an exercise in finding inverse transforms. Inverses can be found either by looking them up in a table of transforms, or directly, by computing them using the inverse formulas given in Table I. The direct approach requires a knowledge of complex variables and contour integration, but it gives more insight into the physical meaning of the solutions.* For example, the inverse expressions contain full information about the natural frequencies of the system.

Linear System Output for Aperiodic Input

The output of a linear system for an aperiodic input $x(t)$ is

$$y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} H(s)X(s)e^{st}ds.$$

Here the Laplace transform has been used. $H(s)$ is the system function and $X(s)$ is the transform of the input $x(t)$.

If the one-sided Laplace transform is used, and if Jordan's Lemma is satisfied (usually this means simply that $H(s)X(s) \to 0$ as $s \to \infty$) then

$y(t) = 0,\ t < 0$

$y(t) =$ sum of the residues of the poles of $H(s)X(s)e^{st},\ t > 0.$

The complete response $y(t)$ consists of two parts. One is the *forced response* $y_f(t)$, in which only the input frequencies appear. [The input $x(t)$ is often called the *forcing function.*] The other part of $y(t)$ is the *transient response* $y_{tr}(t)$.

* If a transform has an infinite number of poles (e.g., a discontinuous function, such as ⎍) it may be difficult or impossible to apply the complex-variable formulas. In such cases, time-domain analysis may be simpler than transform methods.

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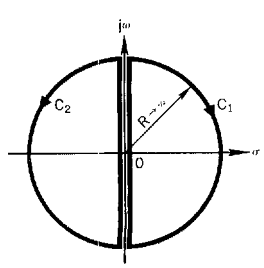
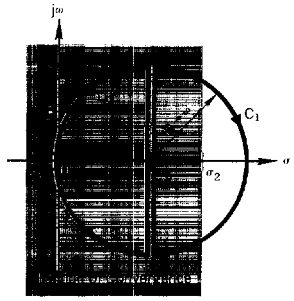
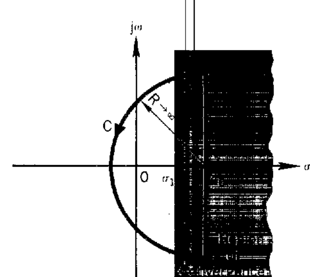
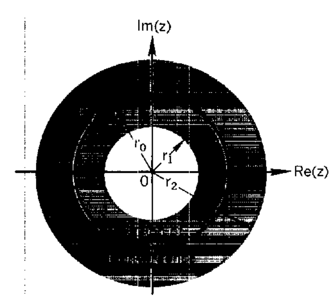
Table I Definitions of Different Transforms with their Inverses

Fourier Transform	Laplace Transform (Two-sided)	Laplace Transform (One-sided)	Z-Transform (Two-sided)
$F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$ $F(j\omega)$ is defined for ω real and exists if $\int_{-\infty}^{\infty} \mid f(t) \mid dt$ exists	$F(s)=\int_{-\infty}^{\infty} f(t)e^{-st}dt$ $s=\sigma+j\omega$ $F(s)$ exists if $\int_{-\infty}^{\infty} \mid f(t)e^{-st} \mid dt$ exists This integral exists in some allowable region of convergence $\sigma_1<\sigma<\sigma_2$	$F(s)=\int_{0}^{\infty} f(t)e^{-st}dt$ $s=\sigma+j\omega$ $F(s)$ exists if $\int_{0}^{\infty} \mid f(t)e^{-st} \mid dt$ exists This integral exists in some allowable region of convergence $\sigma>\sigma_1$	$f^*(t)=\sum_{n=-\infty}^{\infty} f(n)\delta(t-nT)$ $F(z)=\sum_{n=-\infty}^{\infty} f(n)z^{-n}$ or $F(z)=F^*(s)\big\vert_{z=e^{sT}}$ This exists in an allowable annulus of convergence $r_1<\lvert z\rvert<r_2$
$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$ which may be evaluated directly or from tables or $f(t)=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} F(j\omega)e^{j\omega t}d(j\omega)$ $=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} F(s)e^{st}ds\Big\vert_{s=j\omega}$ For $t<0$ $f(t)=\frac{1}{2\pi j}\oint_{C_1} F(s)e^{st}ds$ $=-\left(\text{Sum of residues of poles in right half plane.}\right)$ For $t>0$ $f(t)=\frac{1}{2\pi j}\oint_{C_2} F(s)e^{st}ds$ $=$ Sum of residues of poles in left half plane.	$f(t)=\frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}ds$ $\sigma_1<\sigma_0<\sigma_2$ For $t<0$ $f(t)=\frac{1}{2\pi j}\oint_{C_1} F(s)e^{st}ds$ $=-\left(\text{Sum of residues of poles with real parts to right of } \sigma_0.\right)$ For $t>0$, $f(t)=\frac{1}{2\pi j}\oint_{C_2} F(s)e^{st}ds$ $=$ Sum of residues of poles with real parts to left of σ_0. Note: Residues are defined for counterclockwise contours. Contour C_1 is clockwise. The integral around C_1 is, therefore, *minus* the sum of the residues of the enclosed poles.	$f(t)=\frac{1}{2\pi j}\int_{\sigma_0-j\infty}^{\sigma_0+j\infty} F(s)e^{st}ds$ $\sigma_0>\sigma_1$ For $t>0$ $f(t)=\frac{1}{2\pi j}\oint_{C} F(s)e^{st}ds$ $=$ Sum of residues of poles of $F(s)$.[C encloses all poles of $F(s)$.] Note: As a consequence of the definition of the one-sided transform an acceptable σ_0 is automatically to the right of the poles of $F(s)$.	$f(n)=\frac{1}{2\pi j}\oint_{C} F(z)z^{n-1}dz$ where C is a circle of radius r_0 such that $r_1<r_0<r_2$. For $n\geqq 0$ $f(n)=$ sum of residues of $F(z)z^{n-1}$ inside C. For $n<0$ $f(n)=$ sum of residues of $F(z)z^{n-1}$ outside C. Note: the formula for $f(n)$ is the formula for the coefficients of the Laurent series for $F(z)$ (see Appendix).
			

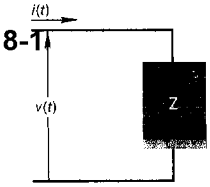
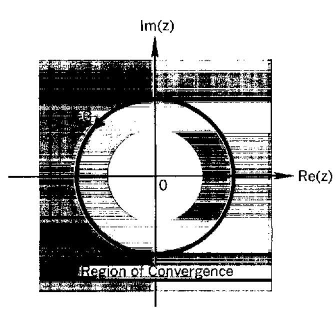


Table III
Impedance Table

Type of Input $i(t)$	Transformed Input	Impedance of Network Element: $Z(s) = \frac{V(s)}{I(s)}$ — R	L	C
i) Damped Sinusoidal $A\, e^{-\sigma_0 t} \cos(\omega_0 t + \phi_0)$	$A\, e^{j\phi_0}$	R	$Ls\big\rvert_{s = -\sigma_0 + j\omega_0}$	$\frac{1}{Cs}\big\rvert_{s = -\sigma_0 + j\omega_0}$
Special Cases a) Sinusoidal, $\sigma_0 = 0$	$A\, e^{j\phi_0}$ phasor notation	R	$Ls\big\rvert_{s = j\omega_0}$	$\frac{1}{Cs}\big\rvert_{s = j\omega_0}$
b) Exponential, $\omega_0 = 0$, $\phi_0 = 0$	A	R	$Ls\big\rvert_{s = -\sigma_0}$	$\frac{1}{Cs}\big\rvert_{s = -\sigma_0}$
c) dc case, $\sigma_0 = 0$, $\omega_0 = 0$, $\phi_0 = 0$	A	R	$Ls\big\rvert_{s = 0}$ short circuit	$\frac{1}{Cs}\big\rvert_{s = 0}$ open circuit
ii) Periodic $\sum_{n=0}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$	Same as case i) using superposition. Each harmonic has an impedance associated with it as in (i) (a).			
iii) Aperiodic $i(t)$	Fourier Transform $I(j\omega) = \int_{-\infty}^{\infty} i(t)\, e^{-j\omega t} dt$ if integral exists	R	$Ls\big\rvert_{s = j\omega}$	$\frac{1}{Cs}\big\rvert_{s = j\omega}$
	Laplace Transform $I(s) = \int_{-\infty}^{\infty} i(t) e^{-st} dt$	R	Ls	$\frac{1}{Cs}$
iv) Sampled $i^*(n)$	System functions for sampled data systems are usually computed first as ratios of Laplace transforms and then converted to Z-transforms.			
v) Random $i(t)$ Autocorrelation Function $= R_i(\tau)$	Power Spectral Density $S_i(j\omega) = \int_{-\infty}^{\infty} R_i(\tau) e^{-j\omega\tau} d\tau$	$\sqrt{R^2}$	$\sqrt{\lvert L^2 s^2 \rvert}\big\rvert_{s = j\omega}$	$\sqrt{\left\lvert \frac{1}{C^2 s^2} \right\rvert}\big\rvert_{s = j\omega}$
		Note: $S_v(j\omega) = \lvert Z(j\omega) \rvert^2 S_i(j\omega)$		

Z-Transform (One-sided)
$f^*(t) = \sum_{n=0}^{\infty} f(n)\delta(t - nT)$ $F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$ or $F(z) = F^*(s)\big\rvert_{z = e^{sT}}$ This exists in an allowable region of convergence outside a circle centered at the origin.
$f(n) = \frac{1}{2\pi j} \oint_C F(z) z^{n-1} dz$ where C encloses all the poles of $F(z)z^{n-1}$ $f(n) =$ sum of the residues of $F(z)z^{n-1}$ Note: the formula for $f(n)$ is the formula for the coefficients of the Laurent series for $F(z)$ (see Appendix).



Only the poles of $X(s)$ contribute to the forced response. The poles of $H(s)$ give the natural frequencies of the system and contribute to the transient response. If we denote the poles of $X(s)$ as $s_1, s_2, \ldots, s_n$ and the poles of $H(s)$ as $S_1, S_2, \ldots, S_m$, and assume zero initial conditions, the response $y(t)$ is

$$y(t) = y_f(t) + y_{tr}(t)$$
$$= \sum_{i=1}^{n} r_i + \sum_{i=1}^{m} R_i$$

where r_i denotes the residue of pole s_i in $H(s)X(s)e^{st}$ and

R_i denotes the residue of pole S_i in $H(s)X(s)e^{st}$.

If the poles of $H(s)$ and $X(s)$ are all *simple* and *distinct*, and if the poles of $H(s)$ do not cancel the zeros of $X(s)$ or vice versa, then all residues will be of the form

$$r_i = \hat{r}_i e^{s_i t} \text{ and } R_i = \hat{R}_i e^{S_i t}$$

See the Appendix or reference 2 for methods of computing residues.

If the initial conditions are not zero the complete response can be obtained to within m arbitrary constants. Again assuming simple, distinct poles and no pole-zero cancellation, $y(t)$ will be

$$y(t) = \sum_{i=1}^{n} r_i + \sum_{i=1}^{m} A_i e^{S_i t}.$$

The constants A_i are determined from the initial conditions, which are usually given as values of $y(t)$ and its derivatives at $t = 0$.

For many forcing functions and for most systems, the forced response $y_f(t)$ is the steady-state response. Examples of aperiodic inputs for which this is true are step functions, ramp functions, and suddenly applied periodic functions.

However, if the time constants of the natural frequencies — poles of $H(s)$ — are greater than the time constants of the input frequencies — poles of $X(s)$ — then the forced response will decay to zero faster than the transient response. In some systems, such as those made up of only L and C, and no R, the transient response will not die out at all. The transient response, therefore, may actually be the total steady-state response. ■

Table II Input-O[illegible]
Linear System. *System Function* $= H(s)$

Type of Input	Transform of Input	Transform of Output	Output
Damped Sinusoidal $x(t) = A\,e^{-\sigma_0 t}\cos(\omega_0 t + \phi_0)$	$X(s) = j\pi A\,[e^{j\phi_0}\delta(s - s_0) + e^{-j\phi_0}\,\delta(s - s_0^*)]$ $s_0 = -\sigma_0 + j\omega_0$	$Y(s) = H(s)X(s)$	$y(t) = \frac{A}{2}[H(s_0)e^{j\phi_0}e^{s_0 t} + H(s_0^*)e^{-j\phi_0}e^{s_0^* t}]$ $= Re[H(s_0)\,A\,e^{j\phi_0}e^{s_0 t}]$
In phasor notation:	$X_T = A\,e^{j\phi_0}$ It is understood that this means $x(t) = Re[A\,e^{j\phi_0}e^{s_0 t}]$	$Y_T = H(s_0)X_T$	$y(t) = Re[H(s_0)X_T e^{s_0 t}]$
Periodic $x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$	Transform of the nth harmonic $X_n(j\omega) = \pi A_n[e^{j\phi_n}\delta(\omega - n\omega_0) + e^{-j\phi_n}\delta(\omega + n\omega_0)]$	By superposition $Y(j\omega) = \sum_{n=0}^{\infty} H(j\omega)\,X_n(j\omega)$	$y(t) = \sum_{n=0}^{\infty} Re[H(jn\omega_0)\,A_n e^{j\phi_n}e^{jn\omega_0 t}]$
In phasor notation:	$X_n = A_n e^{j\phi_n}$	$Y_n = H(jn\omega_0)X_n$	$y(t) = \sum_{n=0}^{\infty} Re[H(jn\omega_0)\,X_n e^{jn\omega_0 t}]$
Aperiodic $x(t)$	$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ if integral exists, or $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$	$Y(j\omega) = H(j\omega)X(j\omega)$ or $Y(s) = H(s)X(s)$	$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[H(j\omega)\,X(j\omega)]e^{j\omega t}d\omega$ or $y(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}[H(s)X(s)]e^{st}ds$
Sampled $x^*(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT)$	$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$	$Y(z) = H(z)X(z)$ where $H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$ where $h(n)$ = unit pulse response	$y(n) = \frac{1}{2\pi j}\oint_C H(z)X(z)z^{n-1}dz$
Random $R_x(\tau)$ = Input Autocorrelation Function	Input Power Spectral Density $S_x(j\omega) = \int_{-\infty}^{\infty} R_x(\tau)e^{-j\omega\tau}d\tau$	$S_y(j\omega) = \lvert H(j\omega)\rvert^2\,S_x(j\omega)$	Output Autocorrelation Function $R_y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\,\lvert H(j\omega)\rvert^2 S_x(j\omega)]\,e^{j\omega\tau}d\omega$

Appendix. Complex-Variable Formulas

1. $\oint_C f(z)dz = 0$ if $f(z)$ is analytic on and inside the closed contour C.

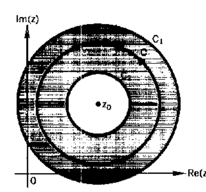
2. $\oint_C g(z)dz = 2\pi j \sum$ [residues of poles of $g(z)$ inside C].

   If $g(z)$ has an nth-order pole at z_0 then

$$\text{residue of } z_0 = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left\{(z - z_0)^n g(z)\right\}\right]_{z = z_0}$$

3. If $f(z)$ is analytic on C_1 and C_2 and in the region between them, then $f(z)$ can be represented in that region by a Laurent Series.

$$f(z) = \sum_{n=-\infty}^{\infty} A_n(z - z_0)^n$$



$$\text{where } A_n = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

C is any curve between C_1 and C_2 and enclosing z_0.

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PILOT SEQUENCE

The course sequence which was chosen was as follows:

I Vectorial Representation of Variables: matrix formats; manipulations; vectorial products; orthogonality; independence; Fourier Series; Laplace representation; convolution; Walsh Functions.

The intent here was to develop a base communications in the course, to set the context of terminology and to introduce the sequence to a group which had indicated strength in the topical area. Applications were treated, homework solutions and several representative journal reprints were distributed through the month, between ensuing sessions.

II Linear System Variables: convolution; Laplace manipulations; applications to linear differential equations; damping considerations; impulse responses; system flow diagram; Z Transforms; sampling; numerical methods: Gauss' elimination, matrix inversion.

The goal here was to backtrack into the previous session, held a month previously and to apply the earlier developed tools to simple linear systems. Some linearization schemes were rationalized; sample applications were treated in class to varying depths, generally on a deterministic basis.

III Probability and Statistics: concepts of discrete and continuous variables; sample space; union; intersection; independence; definitions; density function; distribution function; expectancy operator; moments; confidence limits.

It was hoped here to develop tools for treating probabilistic problems. The attempt was to tie in the discrete abstract variable to several physical situations. Applications were framed to repeat the use of material of the sessions.

IV Stochastic Processes: stationary processes; approximations to Gaussian; filtering and averaging; correlation; convolution; cross-correlation; covariance matrix; power spectral estimates; band limiting effects.

The intent in this session was to relate single continuous variables to the array of tools available to handle generalized data bases. Points of relevance were made to tie in the preceeding sessions to space-time variables found in a number of disciplines. Experimental data was developed in handouts and related to different distributions for signal and noise.

V. Stochastic Processes: general review and excercise of modelling tools presented to date; concepts of signals and interference; properties of space and time variables in single dimension case; conditional probability.

Feedback at this point showed that the pace of preceeding sessions was too fast. It was attempted to recapitulate cumulative material.

VI Detector Subsystems: one dimensional signal and noise; detection; decision threshold; optimum processing; receiver operating characteristics; interference effects from ambient noise, system noise, doppler, reverberation, channel uncertainty in a variety of applications.

It had been hoped here that a consistent approach on a set of commonality subsystem functions could be made for ensuing sessions. The detection function is the most common across a variety of disciplings with applications examples in biomedicine, radar, communications, acoustics, optics, and in seismics.

VII Detector Subsystems: optimum detection; prewhitening; Markov noise; detectability criteria; coherent processing; energy detection; confidence measures; Students' t Test.

Continued work on detection functions.

VIII Space-Time Processing Subsystems: multisensor arrays; signal and noise matrices; prewhitening; matched filters; detection; averaging schemes.

The linear array and its variations was the central model for two sessions on spatial subsystems. This had been cited as an area requiring emphasis earlier.

IX Spatial Processors: optimal arrays; lobes in time and space; cohertency; detectability for several configuarations; near field/far field considerations; non-planar wavefronts.

Intent here was to bring in the cumulative set of modeling tools to a group of spatial applications.